MODELING OF UNIDIRECTIONAL SHORT-FIBER REINFORCED CONCRETE

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Abstract. The interaction between short fibers and concrete in the post-cracking phase influences crack spacing and width in the composite. In order to perform analysis of deformation of a composite and the fiber displacement at the crack, single fiber was examined. A two-fiber model considering the distribution of fiber length and incomplete bonding was developed. Numerical analysis reveals that two-fiber model analysis is believed to be more accurate than that obtained from the single-fiber analysis. Comparing the solution of the single-fiber and two-fiber system shows that the latter gives a greater fiber displacement at the crack. The study was performed in the Department of Structural Engineering, year 2009/2010.
Key words: stress distribution, incomplete bonding, equilibrium, strain difference.

Introduction

In short-fiber reinforced concrete, the fibers are effective in the post-cracking case. The cracking mode of the composite under tensile forces depends on whether the fibers at the crack can transfer the total load needed for crack formation. In the case of short fibers, the multiple cracking can be achieved if the fiber volume content is sufficiently high. In a multiple cracking stage, the concrete is divided into segments of similar length. The force transferring mechanism from the fiber to the concrete and from the concrete to the other fiber has a considerable influence on crack spacing and width.

In general case, fiber may cover two or more cracks if the crack spacing is sufficiently low. Because the matrix stress is zero at the crack, the shear stress between the fiber and concrete is not uniformly distributed across the crack. Therefore the boundary conditions must be stated for the segment between the cracks. With short fibers the fiber's shorter embedded length at the crack may be too small to transfer the load equal to that at the adjacent crack. Thus, the fiber forces at the two cracks are not in balance. It is assumed that at the adjacent crack there is another fiber located symmetrically with respect to the fiber being analysed. The forces in the two-fiber system at the cracks satisfy the equilibrium condition.

In this study by using experimental bond stress-slip relationships (Tepfers, 1973), the stress state in straight short-fiber reinforcement is examined. The reinforcing fibers are interconnected with concrete not through their entire side surfaces. The incomplete bonding can be formed as a result of corrosion of components of the composite or micro voids formed in casting process, etc. In the calculation model it is assumed that the composite consists of two unidirectionally oriented fibers.

The aims of the study are to analyze stress state and deformations of orientated short-fibre reinforced concrete using the proposed single-fibre model, consider incomplete bonding between fibre and concrete, due to various imperfections, develop a model that takes into account stress distribution to an adjacent fibre, and to compare the solutions of single-fibre and two-fibre models.

Materials and Methods

The study was performed in the Department of Structural Engineering, Latvia University of Agriculture, year 2009/2010, supported by ESF, contract No. 2009/0180/1DP/1.1.2.1.2/09/IPIA/VIAA/017.

Single fiber model

In the model it is assumed that the calculation element consists of a fiber with length l_f, crack spacing or embedded length $l_{crc} = l_f/2$, and cross-sectional area A_f included in a concrete (matrix). Here and below, the index *f* refers to fiber, and index *m* – to matrix. It is assumed that in the fiber at the crack act forces P_1 and P_2 (Fig. 1). As the basis of the calculation model of the material, the model given by Cox (Cox, 1952) is used. According to this model, the following linear inhomogeneous differential equation can be written for the action of a tensile force *F* on a particular fiber:

$$\frac{d^2F(x)}{dx^2} = C\left[\frac{F(x)}{E_f A_f} - \varepsilon_m\right], \qquad (1)$$

where *x* is the coordinate in the direction of the fiber; E_f and ε_m are the modulus of elasticity of the fiber and deformation of the matrix, respectively. The constant *C* is determined approximately by using matrix shear modulus G_m and fiber diameter d_f:

$$C = \frac{\pi d_f G_m}{R}, \qquad (2)$$

where 2*R* is medium distance between axes of adjacent fibers.

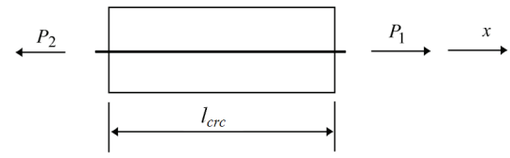


Figure 1. Single-fiber model.

The solution of equation (1) can be written in the form:

$$F(x) = C_1 e^{Bx} + C_2 e^{-Bx} + E_f A_f \varepsilon_m , \quad (3)$$

where $B = \sqrt{\dfrac{C}{E_f A_f}}$.

Assuming that $P_2 = 0$, the unknown constants C_i can be determined from the following boundary conditions:

$$F(0) = 0; \quad F(l_f / 2) = F .$$

In the case of ideal bond between fiber and matrix, the stress distribution through the length of a fiber, when (3) is taken into account, is:

$$\sigma_f(x) = E_f \varepsilon_m \left[1 + \frac{e^{B(x-l_f/2)} - e^{-B(x-l_f/2)}}{e^{Bl_f/2} - e^{-Bl_f/2}} \right] =$$

$$= E_f \varepsilon_m \left[1 + \frac{\mathrm{sh}B(x - l_f/2)}{\mathrm{sh}Bl_f/2} \right] . \quad (4)$$

The length-averaged stress in the fiber is found by integrating (4):

$$< \sigma_f >= \int_0^{l_f/2} \frac{\sigma_f(x)}{l_f} dx = E_f \varepsilon_m \left(1 - \frac{\mathrm{th}Bl_f/4}{Bl_f/2} \right) . \quad (5)$$

By using the rule of mixtures, the modulus of elasticity of a unidirectional reinforced element (composite) in direction *x* can be written as

$$E_x^c = \hat{E}_f \qquad V_f + E_m(1 - V_f) , \quad (6)$$

where V_f and E_m are the volume content of reinforcement and the modulus of elasticity of the matrix, respectively. The effective value of modulus $\hat{E}_f$ can be found by using the deformation in the middle of the fiber, and it takes into account the finite length of fiber (or embedded length):

$$\hat{E}_f \asymp \ < \sigma_f > / \varepsilon_m . \quad (7)$$

The results of studies of the bond properties are presented by F. H. Rabinovich (Рабинович, 2004). On the basis of the mentioned experimental relationships for variation of shear (bond) modulus versus deformation is fixed.

The model developed has been used to study the effect of the finite element length on the modulus of elasticity in fiber direction of themonotropic element. Note that the effect is negligible when $l_f / d_f > 100$, but in the case of a small length ($l_f / d_f < 20$) the difference in magnitudes of the modulus of the monotropic element with continuous fiber and fiber of finite length is significant (Clyne, 2000).

In order to perform analysis of deformation of a composite at the crack, the stress-strain relationship in the case of a single fiber is examined. The shear stress between the fiber and the matrix is expressed as

$$\tau(x) = \frac{1}{\pi d_f} \frac{dF(x)}{dx} \quad (8)$$

and the local strain difference between the fiber and the matrix can be written as:

$$\varepsilon_m(x) - \varepsilon_f(x) = \frac{r}{\pi d_f G_m} \frac{d^2 F(x)}{dx^2} . \quad (9)$$

The fiber displacement at the crack can be derived using (8) in the following way:

$$\Delta = \frac{r}{\pi d_f G_m} \frac{dF(x)}{dx}\Big|_{x=l} = \frac{r P_1 B}{\pi d_f G_m} \left(C_1 e^{Bl} - C_2 e^{-Bl} \right) . \quad (10)$$

Incomplete bond model

The reinforcing fibers are interconnected not through their entire side surfaces. The incomplete bonding is formed as a result of corrosion of composite components or under other conditions (Cabrera, 1996; Schiessl and Reuter, 1992).

Examining the stress distribution mode through the length of a fiber in the case of partial bond, the fiber is arbitrarily divided into sections in which an ideal bond exists between fibers and the ones without bonding. Thus, the regions with ideal contact are concentrated in annular volumes covering the reinforcing fiber (Fig. 2). Curve 1 in Fig. 2 shows the variation of stress in the case of ideal bond σ_f ; curve 2, obtained as a result of geometric construction, shows the variation of the stress $\tilde{\sigma}_f$, i.e., the stress that arises in the case of alternating regions with and without ideal bond. In Fig. 3 the ratio $\tilde{\sigma}_{f\max} / \sigma_{f\max}$ as a function of crack length l_{crc} is shown.

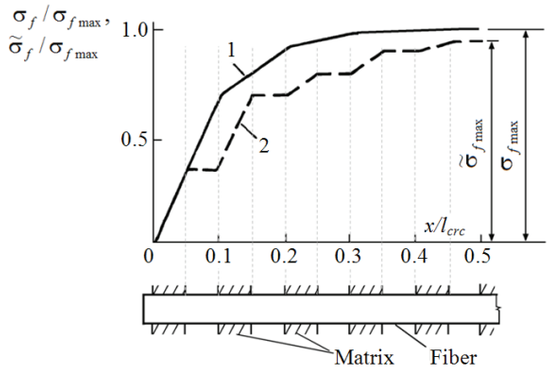


Figure 2. Stress distribution in fiber (l_f = 30 mm) with ideal (1) and incomplete (2) bonding, and the diagram of arbitrary distributed matrix sections with complete bonding.

On the basis of equation (4), the variation of stress $\tilde{\sigma}_f$ along the particle can be expressed as

$$\tilde{\sigma}_f(x) = E_f \varepsilon_m \left[\frac{\tilde{\sigma}_f}{\sigma_f}(l_f) + \frac{\mathrm{sh}B\,(x - l_f/2)}{\mathrm{sh}(B l_f/2)} \right]. \qquad (11)$$

After averaging of stress $\tilde{\sigma}_f$ and considering that effective stiffness of fiber depends on relation $<\tilde{\sigma}_f>/\varepsilon_m$, the reduction factor ψ which takes into account the influence of fiber length l_f and incomplete bonding on the modulus of a composite can be determined by using expression

$$\psi = \frac{\int_{l_{f\min}}^{l_{f\max}} \left[\frac{\tilde{\sigma}_f}{\sigma_f}(l_f) - \frac{\mathrm{tg}(Bl_f/4)}{Bl_f/2} \right] \varphi(l_f)\,dl_f}{\int_{l_{f\min}}^{l_{f\max}} \varphi(l_f)\,dl_f}. \qquad (12)$$

Here function $\varphi(l_f)$ shows distribution of the relative content of fibers as a function of the fiber length.

Assuming the incomplete bonding and distribution of fibers with respect to crack spacing (or embedded length), the modulus of elasticity of a unidirectional reinforced composite E_x^c can be expressed as

$$E_x{}^c = \hat{E}_f \psi V_f + E_m (1 - V_f). \qquad (13)$$

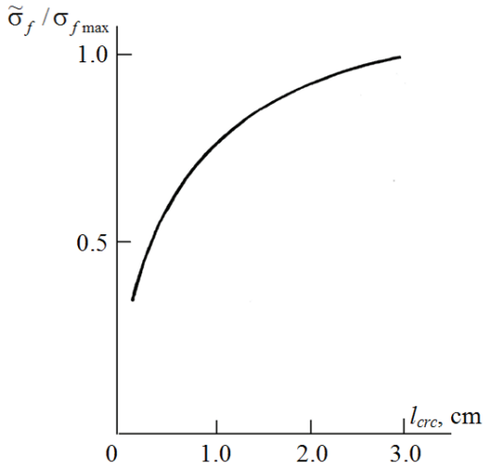


Figure 3. Relationship of relative stress in the fiber of incomplete bonding versus fiber length.

Interaction model of two-fiber system

In the case of two adjacent fibers that are symmetrically located so that the force balance is satisfied at the crack (Fig. 4), the equilibrium condition of the composite in the cross-section is:

$$P = \sum_{i=1}^{2} F_i + \sum_{i=1}^{2} T_i = \sum_{i=1}^{2} \left(A_f \hat{E}_f \psi \varepsilon_f \right)_i + \sum_{i=1}^{2} \left(A_m E_m \varepsilon_m \right)_i =$$

$$= A_f \hat{E}_f \psi \sum_{i=1}^{2} (\varepsilon_f)_i + A_m E_m \sum_{i=1}^{2} (\varepsilon_m)_i, \qquad (14)$$

where $P = P_1 + P_2$, F_i and T_i are the tensile forces in the fiber and matrix, respectively. In the two-fiber system the geometry of matrix area A_m is never axisymmetric with respect to the fibers axis and therefore it is difficult to determine exact value of the area (Рабинович, 2004).

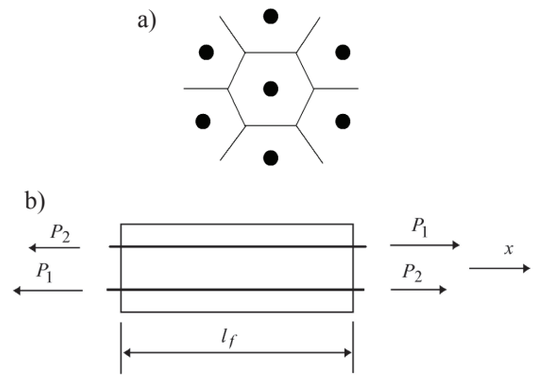


Figure 4. Hexagonal fiber arrangement a) and two-fiber system model b).

Assuming that ε_m is defined as the average of the matrix strain around the two fibers and taking into account the expression (14) it follows that

$$\varepsilon_m = \frac{1}{2A_m E_m} \left[P - A_f \hat{E}_f \psi (\varepsilon_{f1} + \varepsilon_{f2}) \right]. \qquad (15)$$

The local strain difference between, for example, the first fiber and the matrix is:

$$\varepsilon_f(x) - \varepsilon_m(x) =$$

$$= \frac{r}{2A_m E_m} \left[\left(\frac{2A_m E_m}{A_f \hat{E}_f \psi} + 1 \right) F_1(x) - P + F_2(x) \right]. \qquad (16)$$

Combining (9) and (16), the differential equation for $F_1(x)$ can be written

$$\frac{d^2 F_1(x)}{dx^2} = -S\,P + S\,R\,F_1(x) + S\,F_2(x)\ , \qquad (17)$$

where

$$S = \frac{\pi d_f G_m}{2A_m E_m};\ R = \frac{2A_m E_m}{A_f E_f \psi}.$$

Deriving the differential equation for $F_2(x)$ and replacing $F_2(x)$ by its value from the equation (17) we obtain a differential equation of order four for $F_1(x)$:

$$\frac{d^4F_1(x)}{dx^4} - 2S\,R\frac{d^2F_1(x)}{dx^2} + (R^2-1)S^2F_1(x) = \\ = S^2(R-1)P. \quad (18)$$

The solution of equation (18) can be written in the following form:

$$F_1(x) = C_1e^{-B_1x} + C_2e^{B_1x} + C_3e^{-B_2x} + C_4e^{B_2x} + \frac{P}{R+1}, \quad (19)$$

where

$$B_1 = \sqrt{S(R-1)}\,; \quad B_2 = \sqrt{S(R+1)}\,.$$

The force $F_2(x)$ can be obtained from equation (17) as

$$F_2(x) = -C_1e^{-B_1x} - C_2e^{B_1x} + C_3e^{-B_2x} + C_4e^{B_2x} + \frac{P}{R+1}. \quad (20)$$

The unknown constant C_i in expression (20) can be found from boundary conditions

$$F_1(0) = P_1; \quad F_1(l) = P_2; \quad F_2(0) = P_2; \quad F_2(l) = P_1 \quad (21)$$

and taking into account values for $P_i, B_i, R,$ and S. Because of the size of the expressions they are not given here. The fiber displacement at the crack in the case of the two-fiber model is:

$$\Delta = \frac{r}{\pi d_f G_m}\frac{dF_2(x)}{dx}\Big|_{x=l} = \\ = \frac{r}{\pi d_f G_m}\left(B_1C_1e^{-B_1l} - B_1C_2e^{B_1l} - B_2C_3e^{-B_2l} + B_2C_4e^{B_2l}\right). \quad (22)$$

Results and Discussion

In order to compare the results in the case of a single-fiber and two-fiber system, some numerical analyses were performed. The following material parameters were chosen: $d_f = 0.5$ mm, V_f = 4%, E_f = 210000 MPa, E_m = 50000 MPa, G_m = 20000 MPa. The segment length between two cracks (embedded length) is chosen 60 mm, and the fiber forces at the cracks – P_1 = 0.1 kN and P_2 = 0.

The distribution of the fiber force $F(x)$ is shown in Fig. 5. The solid lines represent the two-fiber system and the dashed ones – the single-fiber system. It is seen from Fig. 6 that the fiber force in the single-fiber analysis is twice as that obtained in the two-fiber analysis.

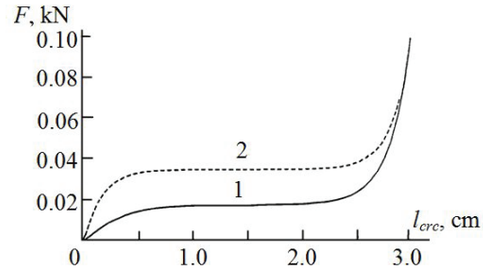


Figure 5. Distribution of fiber (l_{crc} = 30 mm) force: 1 two-fiber system, 2 single-fiber system.

On the basis of the two cases analysed separately (Figs 6 and 7) with P_1 = 0.1 kN, P_2 = 0, and P_1 = 0, P_2 = 0.1 kN it is determined that the total matrix force distributions are equal. The matrix force was derived by expression $T(x) = P - F_1(x) - F_2(x)$.

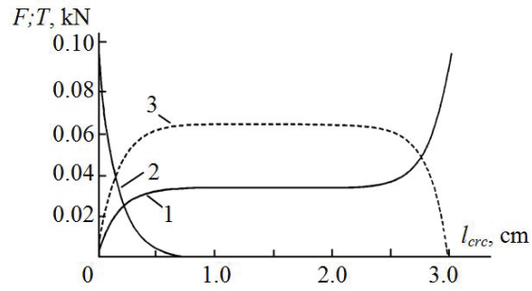


Figure 6. Distribution of fiber force (1, 2) and matrix force (3) in the single-fiber system (l_{crc} = 30 mm, P_1 = 0.1 kN, P_2 = 0).

In the two-fiber system, the fiber force distribution is symmetrical while in a single-fiber analysis it is not symmetrical due to the exceeded boundary conditions (Fig. 6). Line 2 in Fig. 7 shows the distribution of compensation force needed to balance the system. Note that for the single-fiber system the fiber displacement at the crack is Δ = 1.032 x 10^{-3} mm, and for the two-fiber system it is Δ = 1.155 x 10^{-3}, where the relative difference is approximately 12%. Obviously, the two-fiber system analysis gives more accurate results for the fiber displacement at the crack.

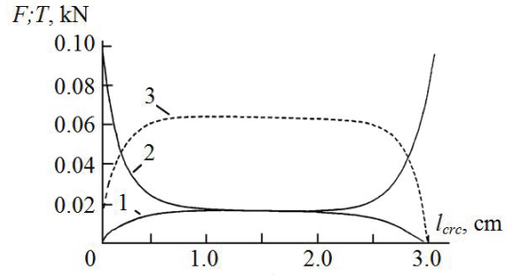


Figure 7. Distribution of fiber force (1, 2) and matrix force (3) in the two-fiber system (l_{crc} = 30 mm, P_1 = 0.1 kN, P_2 = 0, P_1 = 0, P_2 = 0.1 kN).

Conclusions

By using the proposed model, the stress state and deformation caused by axial force in the calculation element with oriented short-fiber reinforcement were studied.

To satisfy the equilibrium conditions at the crack in the case of cracked concrete a two-fiber model is developed.

The model takes into account distribution of fiber length (embedded length) considering incomplete bonding.

The fiber displacement at the crack derived from the two-fiber model analysis is predicted to be more accurate than that obtained from the single-fiber analysis. Comparison of the solutions of the single-fiber and two-fiber systems (with l_f = 60 mm and V_f = 4%) shows that the latter gives a greater fiber displacement at the crack.

In order to apply the proposed model to real cement composites, the fiber forces at the cracks should be derived considering the crack width, and a theory in the multiple cracking cases is still to be developed.

References

1. Cabrera J.G. (1996) Deterioration of concrete due to reinforcement steel corrosion. *Cement and Concrete Composites,* Volume 18, Issue 1. pp. 47-59.
2. Clyne B. (2000) Mechanics of Composite Materials. Available at: http://www.matter.org.uk/matscicdrom/manual/co.html, 20 January 2010.
3. Cox H.L. (1952) The elasticity and strength of paper and other fibrous materials. *British Journal of Applied Physics*, V3, pp. 72-79.
4. Schiessl R., Reuter C. (1992) Bond strength of epoxy-coated reinforcing bars. In: Bond in Concrete. *Proceedings of International Conference From Research to Practice*. Riga Technical University, Riga, pp. 5-20.
5. Tepfers R. (1973) *A theory of bond applied to overlapped tensile reinforcement splices for deformed bars*, Chalmers University of Technology, Göteborg, 328 p.
6. Рабинович Ф.Н. (2004) *Композиты на основе дисперсно армированных бетонов* (Composites based on disperse reinforced concrete), Издательство АСВ, Москва, 560 с. (in Russian).